CENTRAL ELEMENTS OF THE ALGEBRAS $U_q'(\mathrm{so}_m)$ AND $U_q(\mathrm{iso}_m)$

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Abstract

The aim of this paper is to give a set of central elements of the algebras $U_q'(\mathrm{so}_m)$ and $U_q(\mathrm{iso}_m)$ when q is a root of unity. They are surprisingly arise from a single polynomial Casimir element of the algebra $U_q'(\mathrm{so}_3)$. It is conjectured that the Casimir elements of these algebras under any values of q (not only for q a root of unity) and the central elements for q a root of unity derived in this paper generate the centers of $U_q'(\mathrm{so}_m)$ and $U_q(\mathrm{iso}_m)$ when q is a root of unity.

1. The algebra $U_q'(\mathrm{so}_m)$ is a nonstandard q-deformation of the universal enveloping algebra $U(\mathrm{so}_m)$ of the Lie algebra so_m. It was defined in [1] as the associative algebra (with a unit) generated by the elements I_{21}, $I_{32}, \cdots, I_{m,m-1}$ satisfying the defining relations

$$I_{i+1,i}I_{i,i-1}^2-(q+q^{-1})I_{i,i-1}I_{i+1,i}I_{i,i-1}+I_{i,i-1}^2I_{i+1,i}=-I_{i+1,i}, \tag{1}$$

$$I_{i+1,i}^2I_{i,i-1}-(q+q^{-1})I_{i+1,i}I_{i,i-1}I_{i+1,i}+I_{i,i-1}I_{i+1,i}^2=-I_{i,i-1}, \tag{2}$$

$$[I_{i,i-1},I_{j,j-1}]=0 \quad \text{for} \quad |i-j|>1, \tag{3}$$

where $[.,.]$ denotes the usual commutator. In the limit $q\to 1$ formulas (1)–(3) give the relations defining the universal enveloping algebra $U(\mathrm{so}_m)$. Note also that the relations (1) and (2) differ from the q-deformed Serre relations in the approach of Drinfeld and Jimbo to quantum orthogonal algebras (see, for example, [2]) by presence of nonzero right hand sides in (1) and (2).

For the algebra $U_q'(\mathrm{so}_3)$ the relations (1)–(3) are reduced to the following two relations:

$$I_{32}I_{21}^2-(q+q^{-1})I_{21}I_{32}I_{21}+I_{21}^2I_{32}=-I_{32}, \tag{4}$$

$$I_{32}^2I_{21}-(q+q^{-1})I_{32}I_{21}I_{32}+I_{21}I_{32}^2=-I_{21}. \tag{5}$$

Denoting I_{21} and I_{32} by I_1 and I_2, respectively, and introducing the element $I_3:=q^{1/2}I_1I_2-q^{-1/2}I_2I_1$ we find that relations (4) and (5) are equivalent to three relations

$$q^{1/2}I_1I_2-q^{-1/2}I_2I_1=I_3, \tag{6}$$

$$q^{1/2}I_2I_3-q^{-1/2}I_3I_2=I_1, \tag{7}$$

$$q^{1/2}I_3I_1-q^{-1/2}I_1I_3=I_2. \tag{8}$$

The Inonu–Wigner contruction applied to the algebra $U_q'(\mathrm{so}_m)$ leads to the algebra $U_q(\mathrm{iso}_{m-1})$ which was defined in [3]. The algebra $U_q(\mathrm{iso}_m)$ is the associative algebra (with a unit) generated by the elements $I_{21}, I_{32}, \cdots, I_{m,m-1}, T_m$ such that the elements $I_{21}, I_{32}, \cdots, I_{m,m-1}$ satisfy the defining relations of the algebra $U_q'(\mathrm{so}_m)$ and the additional defining relations

$$I_{m,m-1}^2 T_m - (q+q^{-1}) I_{m,m-1} T_m I_{m,m-1} + T_m I_{m,m-1}^2 = -T_m,$$

$$I_{m,m-1} T_m^2 - (q+q^{-1}) T_m I_{m,m-1} T_m + T_m^2 I_{m,m-1} = -I_{m,m-1},$$

$$[I_{k,k-1}, T_m] := I_{k,k-1} T_m - T_m I_{k,k-1} = 0 \quad \text{if} \quad k < m.$$

If $q = 1$, then these relations define the universal enveloping algebra $U(\mathrm{iso}_m)$ of the Lie algebra iso_m of the Lie group $ISO(m)$.

The algebra $U(\mathrm{iso}_2)$ is generated by two elements I_{21} and T_2 satisfying the relations

$$I_{21}^2 T_2 - (q+q^{-1}) I_{21} T_2 I_{21} + T_2 I_{21}^2 = -T_2, \tag{9}$$

$$I_{21} T_2^2 - (q+q^{-1}) T_2 I_{21} T_2 + T_2^2 I_{21} = -I_{21}. \tag{10}$$

Denoting I_{21} by I and introducing the element $T_1 := [I, T_2]_q \equiv q^{1/2} I T_2 - q^{-1/2} T_2 I$, we find that the relations (9) and (10) are equivalent to the relations

$$[I, T_2]_q = T_1, \quad [T_1, I]_q = T_2, \quad [T_2, T_1]_q = 0. \tag{11}$$

2. In $U_q'(\mathrm{so}_m)$ we can determine [4] elements analogous to the basis matrices I_{ij}, $i > j$, (defined, for example, in [5]) of so_m. In order to give them we use the notation $I_{k,k-1} \equiv I^+_{k,k-1} \equiv I^-_{k,k-1}$. Then for $k > l+1$ we define recursively

$$I^{\pm}_{kl} := [I_{l+1,l}, I_{k,l+1}]_{q^{\pm 1}} \equiv q^{\pm 1/2} I_{l+1,l} I_{k,l+1} - q^{-\pm 1/2} I_{k,l+1} I_{l+1,l}. \tag{12}$$

The elements I^+_{kl}, $k > l$, satisfy the commutation relations

$$[I^+_{ln}, I^+_{kl}]_q = I^+_{kn}, \quad [I^+_{kl}, I^+_{kn}]_q = I^+_{ln}, \quad [I^+_{kn}, I^+_{ln}]_q = I^+_{kl} \quad \text{for} \quad k > l > n, \tag{13}$$

$$[I^+_{kl}, I^+_{nr}] = 0 \quad \text{for} \quad k > l > n > r \ \text{ and } \ k > n > r > l, \tag{14}$$

$$[I^+_{kl}, I^+_{nr}]_q = (q - q^{-1})(I^+_{lr} I^+_{kn} - I^+_{kr} I^+_{nl}) \quad \text{for} \quad k > n > l > r. \tag{15}$$

For I^-_{kl}, $k > l$, the commutation relations are obtained by replacing I^+_{kl} by I^-_{kl} and q by q^{-1}.

Using the diamond lemma (see, for example, Chapter 4 in [2]), N. Iorgov proved the Poincaré–Birkhoff–Witt theorem for the algebra $U_q'(\mathrm{so}_m)$ (proof of it will be published):

Theorem 1. *The elements* ${I^+_{21}}^{n_{21}} {I^+_{32}}^{n_{32}} {I^+_{31}}^{n_{31}} \cdots {I^+_{m1}}^{n_{m1}}$, $n_{ij} = 0, 1, 2, \cdots$, *form a basis of the algebra* $U_q'(\mathrm{so}_m)$.

This theorem is true if the elements I^+_{ij} are replaced by the corresponding elements I^-_{ij}.

Using the generating elements $I_{21},\ I_{32},\cdots,I_{m,m-1}$ of the algebra $U_q(\mathrm{iso}_m)$ we define by formula (12) the elements $I^{\pm}_{ij}$, $i>j$, in this algebra. Besides, in $U_q(\mathrm{iso}_m)$ we also define recursively the elements

$$T^{\pm}_k := [I_{k+1,k}, T^{\pm}_{k+1}]_{q^{\pm}}, \quad k=1,2,\cdots,m-1.$$

It is shown in [6] that the elements I^{+}_{ij}, $i>j$, and T^{+}_k, $1\le k\le m$, satisfy the commutation relations (13)–(15) and the relations

$$[I^{+}_{ln}, T^{+}_l]_q = T^{+}_n, \quad [T^{+}_n, I^{+}_{ln}]_q = T^{+}_l \quad \text{for} \quad l>n,$$

$$[T^{+}_l, I^{+}_{np}] = 0 \quad \text{for} \ \ l>n>p \ \ \text{or} \ \ n>p>l,$$

$$[T^{+}_l, I^{+}_{np}] = (q-q^{-1})(T^{+}_n I^{+}_{lp} - T^{+}_p I^{+}_{nl}) \quad \text{for} \quad n>l>p,$$

$$[T^{+}_l, T^{+}_n]_q = 0 \quad \text{for} \quad n<l.$$

For $U_q(\mathrm{iso}_m)$ the Poincaré–Birkhoff–Witt theorem is formulated as

Theorem 2. *The elements* ${I^{+}_{21}}^{n_{21}} {I^{+}_{32}}^{n_{32}} {I^{+}_{31}}^{n_{31}} \cdots {I^{+}_{m1}}^{n_{m1}} {T^{+}_1}^{n_1} {T^{+}_2}^{n_2} \cdots {T^{+}_m}^{n_m}$ *with* $n_{ij}, n_k = 0,1,2,\cdots$, *form a basis of the algebra* $U_q(\mathrm{iso}_m)$.

3. It is easy to check that for any value of q the algebra $U'_q(\mathrm{so}_3)$ has the Casimir element

$$C_q = q^2 I_1^2 + I_2^2 + q^2 I_3^2 + q^{1/2}(1-q^2) I_1 I_2 I_3.$$

As in the case of quantum algebras (see, for example, Chapter 6 in [2]), at q a root of unity this algebra has additional central elements.

Theorem 3. *Let* $q^n=1$ *for* $n\in\mathbb{N}$ *and* $q^j\ne 1$ *for* $0<j<n$. *Then the elements*

$$C^{(n)}(I_j) = \sum_{j=0}^{[\frac{n-1}{2}]} \binom{n-j}{j} \frac{1}{n-j} \Big(\frac{i}{q-q^{-1}}\Big)^{2j} I_j^{n-2j}, \quad j=1,2,3, \tag{16}$$

belong to the center of $U'_q(\mathrm{so}_3)$, *where* $[x]$ *for* $x\in\mathbb{R}$ *denotes the integral part of* x.

The proof of this theorem is rather complicated (see [7]). First it is proved that $C^{(n)}(I_1)$ belongs to the center of $U'_q(\mathrm{so}_3)$. This proof is based on the formula $I_3 I_1^m = p_m(I_1) I_2 + q_m(I_1) I_3$, where

$$p_m(x) = q^{-\frac12} (\tfrac{x(q+q^{-1})}{2})^{m-1} \sum_{t=0}^{[\frac{m-1}{2}]} \tbinom{m}{2t+1} ((\tfrac{q-q^{-1}}{q+q^{-1}})^2 - (\tfrac{2}{x(q+q^{-1})})^2)^t,$$

$$q_m(x) = -q^{\frac12} \tfrac{x(q-q^{-1})}{2} p_m(x) + (\tfrac{x(q+q^{-1})}{2})^m \sum_{t=0}^{[\frac{m}{2}]} \tbinom{m}{2t} ((\tfrac{q-q^{-1}}{q+q^{-1}})^2 - (\tfrac{2}{x(q+q^{-1})})^2)^t.$$

The proof also needs deep combinatorial identities, such that

$$\sum_{t=0}^{[\frac{N-1}{2}]} \tbinom{N}{2t+1} \tbinom{[\frac{N-1}{2}]-t}{[\frac{N-1}{2}]-C} \tbinom{t}{M} =$$

$$= 4^{C-M}\binom{C}{M}(N-2C(1-N'))\frac{(2[\frac{N}{2}])!([\frac{N}{2}]+C-M)!([\frac{N}{2}]-M)!}{[\frac{N}{2}]!(2C+1)!(2[\frac{N}{2}]-2M)!([\frac{N}{2}]-C)!},$$

$$\sum_{j=0}^{d}\binom{n-j}{j}\frac{2d-2j+1+(n-2d-1)^{n'}}{n-j}\binom{[\frac{n-1}{2}]-d}{c-j}\cdot\frac{(2[\frac{n}{2}]-2j)!(n-1-c-d)!([\frac{n}{2}]-c)!}{([\frac{n}{2}]-j)!(d-j+1-n')!(2[\frac{n}{2}]-2c)!(n-2d+n'-1)!} =$$

$$= \frac{\binom{n-1-d}{d}\binom{n-1-c}{c}}{\left(\frac{n}{2}\right)^{1-n'}(n-2d)^{n'}},$$

where $N, n \in \mathbb{N}$, $0 \leqq C, M \leqq [\frac{N-1}{2}]$, $0 \leqq c, d \leqq [\frac{n-1}{2}]$ and $N', n' = 0$ or 1 such that $N' = N \bmod 2$ and $n' = n \bmod 2$. One also needs an extensive use of the fact that q is a root of unity.

If it is proved that $C^{(n)}(I_1)$ belongs to the center of $U'_q(\mathrm{so}_3)$, then we have to use the automorphism ρ: $U'_q(\mathrm{so}_3) \to U'_q(\mathrm{so}_3)$ defined by relations $\rho(I_1) = I_2$, $\rho(I_2) = I_3$, $\rho(I_3) = I_1$. This automorphism shows that $C^{(n)}(I_2)$ and $C^{(n)}(I_3)$ also belong to the center of $U'_q(\mathrm{so}_3)$.

Conjecture 1. *If q is a root of unity as above, then the elements C_q and $C^{(n)}(I_j)$, $j = 1, 2, 3$, generate the center of $U'_q(\mathrm{so}_3)$.*

4. Central elements of the algebra $U'_q(\mathrm{so}_m)$ for any value of q are found in [8] and [9]. They are given in the form of homogeneous polynomials of elements of $U'_q(\mathrm{so}_m)$. If q is a root of unity, then (as in the case of quantum algebras) there are additional central elements of $U'_q(\mathrm{so}_m)$.

Theorem 4. *Let $q^n = 1$ for $n \in \mathbb{N}$ and $q^j \neq 1$ for $0 < j < n$. Then the elements*

$$C^{(n)}(I^+_{kl}) = \sum_{j=0}^{[\frac{n-1}{2}]}\binom{n-j}{j}\frac{1}{n-j}\left(\frac{i}{q-q^{-1}}\right)^{2j} I^{+\,n-2j}_{kl}, \quad k > l, \tag{17}$$

belong to the center of $U'_q(\mathrm{so}_m)$.

Let us prove this theorem for the algebra $U'_q(\mathrm{so}_4)$ (for the general case a proof is the same). This algebra is generated by the elements I_{43}, I_{32}, I_{21}. We introduce the elements $I_{31} \equiv I^+_{31}$, $I_{42} \equiv I^+_{42}$, $I_{41} \equiv I^+_{41}$ defined as indicated above. Then the elements I_{ij}, $i > j$, satisfy the relations

$$[I_{43}, I_{21}] = 0, \quad [I_{32}, I_{31}]_q = I_{21}, \quad [I_{21}, I_{32}]_q = I_{31}, \tag{18}$$

$$[I_{31}, I_{21}]_q = I_{32}, \quad [I_{43}, I_{42}]_q = I_{32}, \quad [I_{32}, I_{43}]_q = I_{42},$$

$$[I_{42}, I_{32}]_q = I_{43}, \quad [I_{31}, I_{43}]_q = I_{41}, \quad [I_{21}, I_{42}]_q = I_{41},$$

$$[I_{41}, I_{21}]_q = I_{42}, \quad [I_{41}, I_{31}]_q = I_{43}, \quad [I_{42}, I_{41}]_q = I_{21},$$

$$[I_{41}, I_{32}] = 0, \quad [I_{43}, I_{41}]_q = I_{31}, \quad [I_{42}, I_{31}] = (q-q^{-1})(I_{21}I_{43} - I_{41}I_{32}).$$

If one wants to prove that an element X belongs to the center of $U'_q(\mathrm{so}_4)$, it is sufficient to prove that $[X, I_{21}] = [X, I_{32}] = [X, I_{43}] = 0$.

Let us consider the element $C^{(n)}(I_{21})$. It belongs to the subalgebra $U'_q(\mathrm{so}_3)$ generated by I_{21}, I_{32} and I_{31}:

$$\begin{array}{ccc} I_{21} & I_{31} & I_{41} \\ & I_{32} & I_{42} \\ & & I_{43} \end{array}$$

It follows from Theorem 3 that $C^{(n)}(I_{21})$ commutes with element I_{32}. Using the first relation in (18) we easily see that $C^{(n)}(I_{21})$ commutes with I_{43} and therefore $C^{(n)}(I_{21})$ belongs to the center of $U'_q(\mathrm{so}_4)$.

Let us consider the element $C^{(n)}(I_{32})$. In $U'_q(\mathrm{so}_4)$ we separate two subalgebras $U'_q(\mathrm{so}_3)$:

$$\begin{array}{ccc} I_{21} & I_{31} & I_{41} \\ & I_{32} & I_{42} \\ & & I_{43} \end{array}$$

From Theorem 3 we have $[C^{(n)}(I_{32}), I_{21}] = [C^{(n)}(I_{32}), I_{43}] = 0$ and $C^{(n)}(I_{32})$ belongs to the center of $U'_q(\mathrm{so}_4)$. A proof that the element $C^{(n)}(I_{43})$ belongs to the center is the same as for $C^{(n)}(I_{21})$.

The elements $C^{(n)}(I_{31})$, $C^{(n)}(I_{42})$ and $C^{(n)}(I_{41})$ belong to the center of $U'_q(\mathrm{so}_4)$ since they belong to the sabalgebras $U'_q(\mathrm{so}_3)$ generated by triplets

$$I_{41}, \quad I_{31}, \quad I_{43} \qquad \text{and} \qquad I_{21}, \quad I_{41}, \quad I_{42}.$$

(Note that $C^{(n)}(I_{31})$ and $C^{(n)}(I_{42})$ commute with I_{42} and I_{31}, respectively, since $I_{42} = [I_{32}, I_{43}]_q$ and $I_{31} = [I_{21}, I_{32}]_q$.) Theorem is proved.

Conjecture 2. *If q is a root of unity as above, then the central elements of [9] and of Theorem 4 generate the center of $U'_q(\mathrm{so}_m)$.*

5. Let us consider the associative algebra $U'_{q,\varepsilon}(\mathrm{so}_3)$ (where $\varepsilon \geq 0$) generated by three generators J_1, J_2, J_3 satisfying the relations:

$$[J_1, J_2]_q := q^{1/2} J_1 J_2 - q^{-1/2} J_2 J_1 = J_3, \quad [J_2, J_3]_q = J_1, \quad [J_3, J_1]_q = \varepsilon^2 J_2.$$

It is easily proved that this algebra is isomorphic to the algebra $U'_q(\mathrm{so}_3)$ and the corresponding isomorphism is uniquely defined by $J_1 \to \varepsilon I_1$, $J_3 \to \varepsilon I_3$, $J_2 \to I_2$. Therefore, the elements

$$\widetilde{C}^{(n)}(J_i, \varepsilon) := n\varepsilon^n C^{(n)}(J_i/\varepsilon), \ i = 1, 3, \qquad \widetilde{C}^{(n)}(J_2, \varepsilon) := C^{(n)}(J_2)$$

belong to the center of $U'_{q,\varepsilon}(\mathrm{so}_3)$ if $q^n = 1$. By means of the contraction $\varepsilon \to 0$ we transform the algebra $U'_{q,\varepsilon}(\mathrm{so}_3)$ into the algebra $U'_q(\mathrm{iso}_2)$. Under this contraction the commutation relations $[\widetilde{C}^{(n)}(J_i, \varepsilon), J_k] = 0$ transform into the relations $[\widetilde{C}^{(n)}(J_i, 0), J_k] = 0$. In other words, we have proved the following

Theorem 5. *Let $q^n = 1$ for $n \in \mathbb{N}$ and $q^j \neq 1$ for $0 < j < n$. Then the elements T_1^n, T_2^n and $C^{(n)}(I)$ belong to the center of the algebra $U'_q(\mathrm{iso}_2)$.*

It was shown in [10] that the element

$$C_q = q^{-1}T_1^2 + qT_2^2 + q^{-3/2}(1-q^2)T_1T_2I$$

is central in $U_q'(\mathrm{iso}_2)$.

Conjecture 3. *If q is a root of unity as above, then the elements C_q, T_1^n, T_2^n and $C^{(n)}(I)$ generate the center of $U_q'(\mathrm{iso}_2)$.*

Using Theorem 5 and repeating the proof of Theorem 4 we prove the following theorem:

Theorem 6. *Let $q^n = 1$ for $n \in \mathbb{N}$ and $q^j \neq 1$ for $0 < j < n$. Then the elements*

$$C^{(n)}(I_{ij}), \quad i > j, \qquad T_j^n, \quad j = 1, 2, \cdots, m,$$

belong to the center of the algebra $U_q'(\mathrm{iso}_m)$.

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